

## DEPENDENCE SPACES

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ABSTRACT. The Steinitz exchange lemma is a basic theorem in linear algebra used, for example, to show that any two bases for a finite-dimensional vector space have the same number of elements. The result is named after the German mathematician Ernst Steinitz.

We present here another proof of the result of N.J.S. Hughes [2] on Steinitz' exchange theorem for infinite bases. In our proof we assume Kuratowski-Zorn Maximum Principle instead of well ordering. We present some examples of dependence spaces of general nature with their possible applications of the result in other as linear or universal algebra domains of mathematical sciences. The lecture was presented on 77th Workshop on General Algebra, 24th Conference for Young Algebraists in Potsdam (Germany) on 21st March 2009.

## 1. INTRODUCTION ON STEINITZ

We recall some facts of Article by: J.J. O'Connor and E.F. Robertson from an internet biography of Steinitz from School of Mathematics and Statistics University of St Andrews, Scotland. The URL of this page is:

<http://www-history.mcs.st-andrews.ac.uk/Biographies/Steinitz.html> .

Born: 13 June 1871 in Laurahtte, Silesia, Germany (now Huta Laura, Poland)

Died: 29 Sept 1928 in Kiel, Germany.

Ernst Steinitz entered the University of Breslau in 1890. He went to Berlin to study mathematics there in 1891 and, after spending two years in Berlin, he returned to Breslau in 1893. In the following year Steinitz submitted his doctoral thesis to Breslau and, the following year, he was appointed Privatdozent at the Technische Hochschule Berlin - Charlottenburg.

The offer of a professorship at the Technical College of Breslau saw him return to Breslau in 1910. Ten years later he moved to Kiel where he was appointed to the chair of mathematics at the University of Kiel.

Steinitz was a friend of Toeplitz. The direction of his mathematics was also much influenced by Heinrich Weber and by Hensel's results on  $p$ -adic numbers in 1899. In [4] interesting results by Steinitz are discussed. These results were given by Steinitz in 1900, when he was a Privatdozent at the Technische Hochschule Berlin - Charlottenburg, at the annual meeting of the Deutsche Mathematiker-Vereinigung in Aachen. In his talk Steinitz introduced an algebra over the ring of integers whose base elements are isomorphism classes of finite abelian groups. Today this is known as the Hall algebra. Steinitz made a number of conjectures which were later proved by Hall.

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Steinitz is most famous for work which he published in 1910. He gave the first abstract definition of a field in *Algebraische Theorie der Krper* in that year. Prime fields, separable elements and the degree of transcendence of an extension field are all introduced in this 1910 paper. He proved that every field has an algebraically closed extension field, perhaps his most important single theorem.

The now standard construction of the rationals as equivalence classes of pairs of integers under the equivalence relation:  $(a, b)$  is equivalent to  $(c, d)$  if and only if  $ad = bc$  was also given by Steinitz in 1910.

Steinitz also worked on polyhedra and his manuscript on the topic was edited by Rademacher in 1934 after his death.

For more information on the role of Steinitz papers consult the book chapter 400 *Jahre Moderne Algebra*, of [4].

## 2. NOTATION

We use the notation of [2]:  $a, b, c, \dots, x, y, z, \dots$  (with or without suffices) to denote the elements of  $\mathbf{S}$  and  $A, B, C, \dots, X, Y, Z, \dots$  for subsets of  $\mathbf{S}$ ,  $\mathbb{X}, \mathbb{Y}, \dots$  denote a family of subsets of  $\mathbf{S}$ ,  $n$  is always a positive integer.

$A + B$  denotes the union of sets  $A$  and  $B$ ,  $A - B$  denotes the difference of  $A$  and  $B$ , i.e. is the set of those elements of  $A$  which are not in  $B$ .

## 3. DEPENDENT AND INDEPENDENT SETS

The following definition is due to N.J.S. Hughes, invented in 1962 in [2]:

**Definition 3.1.** A set  $\mathbf{S}$  is called a *dependence space* if there is defined a set  $\Delta$ , whose members are finite subsets of  $\mathbf{S}$ , each containing at least 2 elements, and if the Transitivity Axiom is satisfied.

**Definition 3.2.** A set  $A$  is called *directly dependent* if  $A \in \Delta$ .

**Definition 3.3.** An element  $x$  is called *dependent on*  $A$  and is denoted by  $x \sim \Sigma A$  if either  $x \in A$  or if there exist distinct elements  $x_0, x_1, \dots, x_n$  such that

$$(1) (x_0, x_1, \dots, x_n) \in \Delta$$

where  $x_0 = x$  and  $x_1, \dots, x_n \in A$

and *directly dependent on*  $(x)$  or  $(x_0, x_1, \dots, x_n)$ , respectively.

**Definition 3.4.** A set  $A$  is called *dependent* if (1) is satisfied for some distinct elements  $x_0, x_1, \dots, x_n \in A$ , and otherwise  $A$  is *independent*.

**Definition 3.5.** If a set  $A$  is *independent* and for any  $x \in \mathbf{S}$ ,  $x \sim \Sigma A$ , i.e.  $x$  is dependent on  $A$ , then  $A$  is called a *basis of*  $\mathbf{S}$ .

### Definition 3.6. TRANSITIVITY AXIOM:

If  $x \sim \Sigma A$  and for all  $a \in A$ ,  $a \sim \Sigma B$ , then  $x \sim \Sigma B$ .

Note, that the following well known properties of independent sets are satisfied:

- (2) Any subset of an independent set  $A$  is independent,
- (3) A basis is a maximal independent set of  $\mathbf{S}$  and vice versa.
- (4) The family  $(\mathbb{X}, \subseteq)$  of all independent subsets of  $\mathbf{S}$  is partially ordered by the set-theoretical inclusion. Shortly we say that  $\mathbb{X}$  is an ordered set (a po-set).
- (5) Any superset of a dependent set of  $\mathbf{S}$  is dependent.

*Proof* (2) Let  $A$  be an independent set and  $B \subseteq A$ . Then  $B$  is independent, as if not, then in  $B$  there is a finite sequence  $b_0, \dots, b_n$  of elements such that  $(b_0, \dots, b_n) \in \Delta$ . But  $(b_0, \dots, b_n) \in A$ , therefore  $A$  is dependent, a contradiction.

(3) Let  $A$  be a basis of  $\mathbf{S}$ , i.e.  $A$  is independent and for each  $x \in \mathbf{S}$ ,  $x \sim \Sigma A$ . Assume that  $A$  is not a maximal independent set of  $S$ , let  $A \subset B$ , with  $A \neq B$ , where  $B$  is independent and let  $b \in B - A$ . We have  $b \sim \Sigma A$ , as  $A$  is a basis, i.e. there is a sequence of elements:  $(x_0 = b, x_1, \dots, x_n)$  with  $x_1, \dots, x_n \in A$ , and such that  $x_0, x_1, \dots, x_n \in \Delta$ . We get that  $B$  is dependent, a contradiction. We conclude that a basis is a maximal independent set in  $\mathbf{S}$ .

Now, let  $A$  be a maximal independent subset of  $\mathbf{S}$ . We show that  $A$  is a basis. We need to show that for every  $x \in \mathbf{S}$ ,  $x \sim \Sigma A$ .

If  $x \in A$ , then  $x \sim \Sigma A$  by Definition. If  $x$  is not included in  $A$ , then the set  $B = A + \{x\}$  is dependent, i.e. there is a sequence  $(x_0, x_1, \dots, x_n)$  of elements of  $B$ , such that  $(x_0, x_1, \dots, x_n) \in \Delta$ . But  $A$  is independent, therefore one of  $x_i$ , say  $x_0$  is in the set  $B - A$ , i.e.  $x_0 = x$  and all  $x_1, \dots, x_n \in A$  (as all  $x_i$  are different, for  $i = 0, 1, \dots, n$ ). We obtain  $x \sim \Sigma A$ . Therefore  $A$  is a basis of  $\mathbf{S}$ .  $\square$

#### 4. EXAMPLES

**Example 4.1.** Consider the two-dimensional vector space  $\mathbf{R}^2$ .

$\Delta$  is defined as follows. A subset  $X$  of  $\mathbf{R}^2$  containing two parallel vectors is always dependent. A finite subset  $X$  of  $\mathbf{R}^2$  with more than 2 elements is always dependent. The one element subset  $\{0\}$  is defined as dependent. Transitivity axiom is satisfied in such dependence space  $\mathbf{S}$ . Independent subsets contain at least two non parallel vectors. Bases are bases of  $\mathbf{R}^2$  in the classical sense.

**Example 4.2.** Let  $\mathbf{K}$  denotes the set of some (at least two) colours,  $\mathbf{S} = C(K)$  is the set of all sequences of  $K$ .  $\Delta = C_{fin}(\mathbf{K})$  be the set containing all finite at least two-element sequences of elements of  $\mathbf{K}$  with at least one repetition of colours. Then the Transitivity Axiom is satisfied for such defined dependence space  $\mathbf{S}$ .

**Example 4.3.** Let  $C(\mathbf{N})$  denotes the set of all sequences of natural numbers and  $\Delta = C_{fin}(\mathbf{N})$  be the set containing all finite at least two-element sequences of elements of  $\mathbf{N}$  with at least one repetition. Then the Transitivity Axiom is satisfied for such defined dependence space  $\mathbf{S}$ .

**Example 4.4.** A graph is an abstract representation of a set of objects where some pairs of the objects are connected by links. The interconnected objects are represented by mathematical abstractions called vertices, and the links that connect some pairs of vertices are called edges. Typically, a graph is depicted in diagrammatic form as a set of dots for the vertices, joined by lines or curves for the edges. For a graph, call a finite subset  $A$  of its vertices directly dependent if it has at least two elements which are connected. Then for a vertex  $x$ , put  $x \sim \Sigma A$  if there exists a link between  $x$  and a vertex  $a$  in  $A$ . Then the transitivity axiom for such a dependence space is satisfied.

#### 5. PO-SET OF INDEPENDENT SETS

Following K. Kuratowski and A. Mostowski [6] p. 241, a po-set  $(\mathbb{X}, \subseteq)$  is called *closed* if for every chain of sets  $\mathbb{A} \subseteq P(\mathbb{X})$  there exists  $\cup \mathbb{A}$  in  $\mathbb{X}$ , i.e.  $\mathbb{A}$  has the supremum in  $(\mathbb{X}, \subseteq)$ .

**Theorem 5.1.** *The po-set  $(\mathbb{X}, \subseteq)$  of all independent subsets of  $\mathbf{S}$  is closed.*

*Proof* Let  $\mathbb{A}$  be a chain of independent subsets of  $\mathbf{S}$ , i.e.  $\mathbb{A} \subseteq P(\mathbb{X})$ , and for all  $A, B \in \mathbb{A}$  ( $A \subseteq B$ ) or ( $B \subseteq A$ ). We show that the set  $\cup \mathbb{A}$  is independent. Otherwise there exist elements  $(x_0, x_1, \dots, x_n) \in \Delta$  such that  $x_i \in \cup \mathbb{A}$ , for  $i = 0, \dots, n$ . therefore there exists a set  $A \in \mathbb{A}$  such that  $x_i \in A$  for all  $i = 0, \dots, n$ .

We conclude that  $A$  is dependent,  $A \in \mathbb{A}$ , a contradiction.  $\square$

## 6. STEINITZ' EXCHANGE THEOREM

A transfinite version of the Steinitz Exchange Theorem, provides that any independent subset injects into any generating subset. The following is a generalization of Steinitz' Theorem proved originally in 1913 and then in [2]–[3]:

**Theorem 6.1.** *If  $A$  is a basis and  $B$  is an independent subset (of a dependence space  $\mathbf{S}$ ). Then assuming Kuratowski-Zorn Maximum Principle, there is a definite subset  $A'$  of  $A$  such that the set  $B + (A - A')$  is also a basis of  $\mathbf{S}$ .*

*Proof* If  $B$  is a basis then  $B$  is a maximal independent subset of  $\mathbf{S}$  and  $A' = A$  is clear.

Assume that  $A$  is a basis and  $B$  is an independent subset (of the dependence space  $\mathbf{S}$ ). Consider  $\mathbb{X}$  to be the family of all independent subsets of  $\mathbf{S}$  containing  $B$  and contained in  $A + B$ . then  $(\mathbb{X}, \subseteq)$  is well ordered and closed. Therefore assuming Kuratowski-Zorn Maximal Principle [5] there exists a maximal element of  $\mathbb{X}$ .

We show that this maximal element  $X \in \mathbb{X}$  is a basis of  $\mathbf{S}$ .

As  $X \in \mathbb{X}$  then  $B \subseteq X \subseteq A + B$  by the construction. Therefore  $X = B + (A - A')$  for some  $A' \subseteq A$ . We show first that for all  $a \in A$ ,  $a \sim \Sigma X$ . If  $a \in X$  then  $a \sim \Sigma X$  by the definition. If not, then put  $Y = X + \{a\}$ . Then  $X \neq Y$ ,  $X \subseteq Y$ ,  $B \subseteq Y \subseteq A + B$  and  $Y$  is dependent in  $\mathbf{S}$ .

By the definition there exist elements:  $(x_0, x_1, \dots, x_n) \in \Delta$  with  $x_1, \dots, x_n \in X + \{a\}$ , as  $X$  is an independent set. Moreover, one of  $x_i$  is  $a$ , say  $x_0 = a$ . We get  $a \sim \Sigma X$  as  $x_0, x_1, \dots, x_n$  are different.

Now we show that  $X$  is a basis of  $\mathbf{S}$ . Let  $x \in S$ , then  $x \sim \Sigma A$  as  $A$  is a basis of  $\mathbf{S}$ . Moreover for all  $a \in A$ ,  $a \sim \Sigma X$ , thus  $x \sim \Sigma X$  by the Transitivity Axiom.  $\square$

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